$$\min_{G(t^*,\, t_*,\, x_*,\, v^*(\cdot))} \varepsilon^{(m)}(t^*, x) = \varepsilon^{(m+1)}(t_*, x_*) \tag{2.11}$$

that $t^* \geqslant t^\circ$ and $v^*(t) = 2$ almost everywhere on $[t_*, t^*]$, $x^\circ(t^*) \in G(t^*, t_*, x_*, v^*(\cdot))$. But in this case $x^\circ(t^*) > -x_0(t^*)$, $c_0(t^*, x^\circ(t^*)) < c_0(t_*, x_*)$ and (see (2.11)) we have $\varepsilon^{(m+1)}(t_*, x_*) < c_0(t_*, x_*)$ which contradicts the assumption. Thus (see (2.9)) we have proven that

$$E_{m+1} = \{(t, x) : (t, x) \in \Lambda_1, |x| \geqslant a_{m+1}(1 - t)\}$$

Taking into account (2.6) and (2.7) as well as Lemmas 2 and 3, we can show that the following theorem holds.

Theorem. Sets E_k, $k \in N_0$ and E_∞ are defined by the conditions

$$E_k = \{(t, x) : (t, x) \in \Lambda_1, |x| \geqslant a_k(1 - t)\}$$
$$E_\infty = S = \{(t, x) : (t, x) \in [0, 1) \times R^1 \, |x| \leqslant 1 - t\}$$

The author thanks N. N. Krasovskii for constant attention and valuable advice.

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Translated by L. K.

UDC 531.36

STABILITY IN FIRST APPROXIMATION OF STOCHASTIC SYSTEMS WITH AFTEREFFECT

PMM Vol. 40, № 6, 1976, pp. 1116-1121

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(Received December 9, 1975)

The theorem on existence of the Liapunov functionals and the theorem on stability in first approximation for a stochastic differential equation with aftereffect are proved.

The suggestion of the replacement of Liapunov functions by functionals [1] in the investigation of the stability of ordinary differential equations with lag, has been widely utilized in dealing with determinate systems, as well as in the case of linear and nonlinear stochastic systems (see, e. g. [2 — 11]). Results concerning the stability in the first approximation were obtained for stochastic systems in [12 — 18] and others. Use of Liapunov functionals for the differential equations with aftereffect was first encountered in [1, 19, 20] where the inversion theorems were proved and conditions for the stability in first approximation were obtained.

Below a stochastic differential equation with aftereffect is investigated where the random perturbations represent an arbitrary process with independent increments.

Let $\{\Omega, \sigma, \mathbf{P}\}$ be the basic stochastic space and $\{f_t, t \geqslant 0\}$ a monotonously nondecreasing family of σ-algebras $f_t \subset \sigma$; let also θ_t be a family of operators defined by the relation $\theta_t\xi(s) = \xi(t+s)$, where $s \leqslant 0$, $t \geqslant 0$ and $\xi(t)$ is an n-dimensional random process defined on $(-\infty, \infty)$, f_t-measurable when $t > 0$ and f_0-measurable when $t \leqslant 0$; let also $w(t) = (w_1(t), \ldots, w_N(t))$ be an N-dimensional Wiener process, $\nu^\circ(t, A)$ a centered Poisson's measure with the parameter $t\Pi(A)$, the process $w(t)$ and the measure $\nu^\circ(t, A)$ independent of each other and f_t-measurable when $t \geqslant 0$.

Let us consider the following stochastic differential equation:

$$d\xi(t) = a(t, \theta_t\xi)\, dt + \sum_{r=1}^{N} b_r(t, \theta_t\xi)\, dw_r(t) + \int c(u; t, \theta_t\xi)\, \nu^\circ(dt, du) \qquad (1)$$

$$\theta_0\varepsilon = \varphi_0$$

in which $a(t, \varphi)$, $b_r(t, \varphi)$ and $c(u; t, \varphi)$ are vector functionals with values in R^n defined for $t \geqslant 0$, $u \in R^n$ and $\varphi \subset H_0$, H_0 is the set of functions $\varphi(s)$ $(s \leqslant 0)$ with values in R^n, which have left bounds with probability one, are continuous to the right when $s < 0$ and to the left when $s = 0$, and such that

$$\sup_{s\leqslant 0}\mathbf{M}\,|\varphi(s)|^2 < \infty, \quad a(t, 0) \equiv b_r(t, 0) \equiv c(u; t, 0) \equiv 0 \qquad (2)$$

$$|a(t, \varphi)|^2 \leqslant \int_0^\infty |\varphi(-\tau)|^2\, dr_0(t, \tau)$$

$$|b_r(t, \varphi)|^2 \leqslant \int_0^\infty |\varphi(-\tau)|^2\, dr_{1r}(t, \tau)$$

$$|c(u; t, \varphi)|^2 \leqslant \int_0^\infty |\varphi(-\tau)|^2\, dr_2(u; t, \tau)$$

$$r_1(t, \tau) = \sum_{r=1}^{N} r_{1r}(t, \tau), \quad r_2(t, \tau) = \int r_2(u; t, \tau)\, \Pi(du)$$

$$\sup_{t\geqslant 0}\int_0^\infty dr_i(t, \tau) < \infty \quad (i = 0, 1, 2)$$

(d is the sign of differentiation in the last argument).

Equations of this type were studied in a number of papers (e. g. [21, 22]) and the conditions of existence and uniqueness of their solutions obtained. We shall therefore assume these conditions to hold. The nonnegative functional $V(t, \varphi)$ on $[0, \infty) \times H_0$ is such that $V(t, 0) \equiv 0$ and $\lim_{t\to\infty}\mathbf{M}V(t, \theta_t\xi) = 0$, provided that we call $\lim_{t\to\infty}\mathbf{M}\,|\xi(t)|^p = 0$ $(p > 0)$ the F_p-functional.

We also call the function $r(s, \tau)$ $(s \geqslant 0, \tau \geqslant 0)$ a nondecreasing function in τ uniformly integrable if

$$\sup \int_0^\infty \int_{t-\tau}^{t} dr(s+\tau, \tau)\, ds < \infty$$

and if for any $\varepsilon > 0$ we can find T such that

$$\sup \int_T^\infty \int_{t-\tau}^t dr(s+\tau,\tau)\,ds < \varepsilon$$

Here and in what follows, the operation sup is taken over all $t \geqslant 0$.

Note. Using the Ito formula [22] and conditions (2) we can show that the function $\mathbf{M}\,|\,\xi(t)\,|^2$ satisfies the Lipschitz condition. Since a function integrable on $[0, \infty)$ and satisfying the Lipschitz condition tends to zero at infinity, therefore from the condition

$$\int_0^\infty \mathbf{M}\,|\,\xi(t)\,|^2\,dt < \infty \tag{3}$$

it follows that $\lim_{t\to\infty} \mathbf{M}\,|\xi(t)|^2 = 0$.

Theorem 1. Let the conditions (2) and (3) hold and the functions $r_i(t, \tau)$ ($i = 0$, 1, 2) be uniformly integrable. Then an F_2-functional $V(t, \varphi)$ exists such that

$$\mathbf{M}V(t, \theta_t\xi) \geqslant k_1 \mathbf{M}\,|\,\xi(t)\,|^2$$

$$\mathbf{M}V(t, \theta_t\xi) - \mathbf{M}V(0, \varphi_0) \leqslant -k_2 \int_0^t \mathbf{M}\,|\,\xi(s)\,|^2\,ds$$

Proof. The conditions of Theorem 1 are satisfied by the functional

$$V(t, \theta_t\xi) = |\,\xi(t)\,|^2 + r\int_0^\infty |\,\xi(t+s)\,|^2\,ds + \int_0^\infty \int_{t-\tau}^t |\,\xi(s)\,|^2 \left(\sum_{i=0}^2 dr_i(s+\tau, \tau)\right) ds$$

$$r > 2\sqrt{r_0} + r_1 + r_2$$

$$r_i = \sup \int_0^\infty dr_i(t+\tau, \tau) \quad (i = 0, 1, 2)$$

since we can apply to it the Ito integro-differential operator L [22], and

$$LV(t, \theta_t\xi) \leqslant -(r - 2\sqrt{r_0} - r_1 - r_2)\,|\,\xi(t)\,|^2$$

The relation (3) and uniform integrability of the functions $r_i(t, \tau)$ ($i = 0, 1, 2$) imply that it is also an F_2-functional.

Theorem 2. Let a positive definite (i.e. $V(t, \varphi) \geqslant k\,|\,\varphi(0)\,|^\alpha$, $k > 0$, $\alpha > 0$) F_2-functional $V(t, \varphi)$ exist such that

$$\mathbf{M}V(0, \varphi_0) < \infty$$

$$\mathbf{M}\{V(t, \theta_t\xi)/f_s\} - V(s, \theta_s\xi) \leqslant -k\int_s^t \mathbf{M}\{|\,\xi(\tau)\,|^2/f_s\}\,d\tau \quad k > 0,\ t \geqslant s \geqslant 0$$

where $\xi(s)$ is a solution and φ_0 is the initial condition of Eq. (1). Then

$$\mathbf{P}\{\lim_{t\to\infty} \xi(t) = 0\} = 1 \tag{4}$$

Proof. Evidently $V(t, \theta_t\xi)$ is a nonnegative supermartingale, consequently, $\lim V(t, \theta_t\xi)$ exists with probability one [16] and $\mathbf{M}\lim V(t, \theta_t\xi) = \lim \mathbf{M}V(t, \theta_t\xi)$ ($t \to \infty$). The function $\mathbf{M}\,|\,\xi(t)\,|^2$ is integrable on $[0, \infty)$ and (see note) satisfies the Lipschitz condition, therefore $\lim \mathbf{M}\,|\,\xi(t)\,|^2 = 0$. Since $V(t, \varphi)$ is an F_2-functional, we also have $\lim \mathbf{M}V(t, \theta_t\xi) = 0$. From all this it follows that $\mathbf{P}\{\lim V(t, \theta_t\xi) = 0\} = 1$. The relation (4) now follows from the positive definiteness of $V(t, \varphi)$.

Corollary. Let the conditions of Theorem 1 hold. Then the solution of (1) satisfies the condition (4).

To prove it we observe that the conditions of Theorem 1 ensure the existence of a functional satisfying the conditions of Theorem 2. We shall show that the solution of (1) satisfies the condition (4) even in the case when the conditions of Theorem 1 hold not for (1), but for its first order approximation, i.e. for a linear equation with the coefficients sufficiently close to the coefficients of (1).

Let us consider the equation

$$d\xi(t)=\int_0^\infty dA(t,\tau)\xi(t-\tau)\,dt+\sum_{r=1}^{N}\int_0^\infty dB_r(t,\tau)\xi(t-\tau)\,dw_r(t)+ \tag{5}$$
$$\iint_0^\infty dC(u;t,\tau)\xi(t-\tau)\nu^\circ(dt,du)$$

the coefficients of which satisfy the conditions ($\|\cdot\|$ is the operator norm of the matrix)

$$\sup\int_0^\infty \|dA(t,\tau)\|<\infty,\quad \sup\sum_{r=1}^{N}\left(\int_0^\infty \|dB_r(t,\tau)\|\right)^2<\infty \tag{6}$$
$$\sup\int\left(\int_0^\infty \|dC(u;t,\tau)\|\right)^2\Pi(du)<\infty$$

In addition, the functions $p_i(t,\tau)$ $(i=0,1,2)$, where

$$dp_0(t,\tau)=\|dA(t,\tau)\|\int_0^\infty \|dA(t,s)\|$$

$$dp_{1r}(t,\tau)=\|dB_r(t,\tau)\|\int_0^\infty \|dB_r(t,s)\|$$

$$dp_2(u;t,\tau)=\|dC(u;t,\tau)\|\int_0^\infty \|dC(u;t,s)\|$$

$$dp_1(t,\tau)=\sum_{r=1}^{N}dp_{1r}(t,\tau),\quad dp_2(t,\tau)=\int dp_2(u;t,\tau)\,\Pi(du)$$

are uniformly integrable.

Let the condition (3) hold for Eq. (5). Then the functional

$$V_0(t,\theta_t\xi)=|\xi(t)|^2+p\int_0^\infty |\xi(t+s)|^2ds+\int_0^\infty\int_{t-\tau}^{t}|\xi(s)|^2\left(\sum_{i=0}^{2}dp_i(s+\tau,\tau)\right)ds$$
$$p>2\sqrt{p_0}+p_1+p_2$$
$$p_i=\sup\int_0^\infty dp_i(t+\tau,\tau)<\infty\quad (i=0,1,2)$$

is an F_2-functional and $L_0V_0(t,\theta_t\xi)\leqslant -k|\xi(t)|^2$ $(k>0)$, where L_0 is the Ito operator corresponding to Eq. (5).

Let the coefficients of (1) and (5) be connected by the following conditions:

$$|a(t,\varphi)-\int_0^\infty dA(t,\tau)\varphi(-\tau)|\leqslant\gamma\int_0^\infty|\varphi(-\tau)|dq_0(t,\tau) \tag{7}$$

$$\left|b_r(t,\varphi)-\int_0^\infty dB_r(t,\tau)\varphi(-\tau)\right|\leqslant\gamma\int_0^\infty|\varphi(-\tau)|dq_{1r}(t,\tau)$$

$$\left|c(u;t,\varphi)-\int_0^\infty dC(u;t,\tau)\varphi(-\tau)\right|\leqslant\gamma\int_0^\infty|\varphi(-\tau)|dq_2(u;t,\tau)$$

$$dq_1(t,\tau)=\sum_{r=1}^N dq_{1r}(t,\tau),\quad dq_2(t,\tau)=\int dq_2(u;t,\tau)\Pi(du)$$

$$q_0=\sup\int_0^\infty dq_0(t,\tau),\quad q_{1r}=\sup\int_0^\infty dq_{1r}(t,\tau),\quad q_2(u)=\sup\int_0^\infty dq_2(u;t,\tau)$$

and the functions $q_i(t,\tau)$ $(i=0,1,2)$ be uniformly integrable. Consider the functional

$$V_1(t,\theta_t\xi)=V_0(t,\theta_t\xi)+\gamma\int_0^\infty\int_{t-\tau}^t|\xi(s)|^2dm(s+\tau,\tau)ds$$

$$dm(t,\tau)=dq_0(t,\tau)+\frac{1}{2}(dq_1(t,\tau)+dq_2(t,\tau))+\sum_{r=1}^N q_{1r}(dr_{1r}(t,\tau)+dp_{1r}(t,\tau))+$$
$$\int q_2(u)(dr_2(u;t,\tau)+dp_2(u;t,\tau))\Pi(du)$$

$$m_0=\sup\int_0^\infty dm(t+\tau,\tau)$$

and estimate the expression

$$LV_1(t,\theta_t\xi)=L_0V_0(t,\theta_t\xi)+2\left(a(t,\theta_t\xi)-\int_0^\infty dA(t,\tau)\xi(t-\tau),\xi(t)\right)+$$
$$\sum_{r=1}^N\left(|b_r(t,\theta_t\xi)|^2-\left|\int_0^\infty dB_r(t,\tau)\xi(t-\tau)\right|^2\right)+$$
$$\int\left(|c(u;t,\theta_t\xi)|^2-\left|\int_0^\infty dC(u;t,\tau)\xi(t-\tau)\right|^2\right)\Pi(du)+$$
$$\gamma|\xi(t)|^2\int_0^\infty dm(t+\tau,\tau)-\gamma\int_0^\infty|\xi(t-\tau)|^2dm(t,\tau)\leqslant-[k-\gamma(m_0+q_0)]|\xi(t)|^2$$

It follows that for fairly small γ, such $k_1>0$ can be found that $LV_1(t,\theta_t\xi)\leqslant-k_1|\xi(t)|^2$. Moreover, the functional $V_1(t,\varphi)$ is an F_2-functional since $V_0(t,\varphi)$ is an F_2-functional and the function $m(t,\tau)$ is uniformly integrable.

Thus we have proved the following theorem.

Theorem 3. Let the coefficients of Eqs. (1) and (5) satisfy the conditions (2), (6) and (7) (the last one at fairly small γ). Let also the functions $p(t,\tau)$, $q(t,\tau)$ and $r(t,\tau)$ be all uniformly integrable and the solution of Eq. (5) satisfy the condition (3). Then the solution of Eq. (1) satisfies the condition (4).

In conclusion, the author thanks V. B. Kolmanovskii for the interest shown.

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Translated by L. K.

UDC 532.529

ON THE SELF-SIMILAR SOLUTION OF NAVIER-STOKES EQUATIONS WITH VOLUME SOURCES AND SINKS OF MASS

PMM Vol. 40, № 6, 1976, pp. 1121-1124
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(Received January 4, 1975)

Unlike the investigations in [1, 2] of the motion of fluid with surface sources and sinks of mass (injection and suction), the flow is considered here in the presence of uniformly distributed mobile volume sources and sinks in flat and round channels. It is shown that far away from the inlet a self-similar solution of the system of equations of motion can be obtained. The results are applicable, for instance, to two-phase (vapor-liquid) streams with condensation or evaporation for small volume concentrations of the discrete phase and absence of phase slip.

1. The steady axisymmetric flow of fluid in pipes with volume sources or sinks of mass which move at the medium velocity, is defined by the system of equations

$$u_x \frac{\partial u_x}{\partial x} + u_r \frac{\partial u_x}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{r^\alpha}\left[\frac{\partial}{\partial x}\left(r^\alpha \frac{\partial u_x}{\partial x}\right) + \frac{\partial}{\partial r}\left(r^\alpha \frac{\partial u_x}{\partial r}\right)\right] \qquad (1.1)$$

$$u_x \frac{\partial u_r}{\partial x} + u_r \frac{\partial u_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\nu}{r^\alpha}\left[\frac{\partial}{\partial x}\left(r^\alpha \frac{\partial u_r}{\partial x}\right) + \frac{\partial}{\partial r}\left(r^\alpha \frac{\partial u_r}{\partial r}\right) - \left(\frac{u_r}{r}\right)^\alpha\right]$$

$$\frac{\partial}{\partial x}(r^\alpha u_x) + \frac{\partial}{\partial r}(r^\alpha u_r) = -r^\alpha \frac{\varkappa}{\rho}$$

where u_x and u_r are velocity vector components in the longitudinal and radial directions, $\varkappa$ is the capacity of volume sources or sinks ($\varkappa > 0$ related to sinks, $\varkappa < 0$ to sources), $\alpha = 0$ for a flat channel, and $\alpha = 1$ for a round pipe.

Let us consider the case of $\varkappa = \text{const}$. We shall seek a self-similar solution for system (1.1) far from the tube inlet in a form that satisfies the equation of continuity